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International Journal of Heat and Mass Transfer 51 (2008) 3504–3513
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Heat transfer to longitudinal laminar flow between thin cylinders

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Received 25 July 2007
Available online 29 January 2008

Abstract

The forced-convection longitudinal flow of a viscous and heat-conducting fluid in a system of thin parallel cylinders, arranged in regular patterns, is studied. The aim of the study is to determine both the velocity field and temperature field between the system of thin cylinders. A new way of calculating i.e. the use of a complex number is proposed. The final results are analytical formulas for determining the longitudinal permeability and the Nusselt number of the fluid and the system of thin cylinders.

1. Introduction

This paper is a study of the longitudinal flow of fluid between systems of thin and long parallel cylinders arranged parallel in regular patterns with a given packing fraction.

The aim of the study is to determine the forced flow evoked by the longitudinal pressure gradient for a viscous and heat-conducting fluid, in particular, the influence of geometry (cylinder thickness and their arrangement) and of the thermal properties of the fluid.

Problems of the fluid flow between system cylinders are often found in technology as well as in nature. They are found in some nuclear reactors, in heat-transfer equipment [14,15] and also in the process of chemical fibre spinning [9,12,13] or the coating of wires. In nature they take place during air motion in grain fields and forests.

The problem of determining the field of the velocity was solved by Emersleben [1–3] for a square net and different two periodic nets obtained by the superposition of square nets but only for rods of a special profile, resulting from the form assumed by the solution with singularities which are regularly arranged. Sparrow and Loeffler also dealt with this problem [4,5]. In the year 1959 they provided the approximate solution for flow round a triangular and square net of cylinders by the use of the method of boundary collocation. Sparrow and Loeffler in the year 1961 solved the second problem of heat transfer, practising the method of the boundary collocation, for cylinders arranged in a triangular network [5]. Happel [6] solved the problem of flow round nets of cylinders via the cellular method. The numerical solution of the velocity field can be found in works of Schmidt [8] and Szołochow et al. [7]. In their last paper they also solved a problem through electric modeling. In articles [10,11] an analytical solution moved closer to the field of the speed for cylinders very thin, was found by the method of the superposition of solutions.

Yang in papers [14,15] used the method practised by Sparrow and Loeffler for outlining the field of the velocity and the heat of transfer in the flow in which the influence of the buoyancy force to conditions of the flow is considered.

In paper [17] the natural-convection flow of a viscous and heat-conducting fluid in system of cylinders arranged vertically in a regular pattern was studied.

Recently in paper [18] an experimental study of free and forced convective heat transfer along vertical slender cylinders was presented.

The present paper deals with forced flow between parallel cylinder arrays and proposes a new way of calculating by the use of a complex number. In this study the method applied in papers [10,11,16] is the starting point for the present paper. A more comfortable notation of results was introduced with the help of complex numbers. The object of the paper is introduced by the analytical

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0017-9310/$ - see front matter
doi:10.1016/j.ijheatmasstransfer.2007.10.032

Nomenclature

$\boldsymbol{w}$	velocity	m	number of cylinders contained in the group n
W	dimensionless velocity	N	amount of groups held for calculations
T	temperature	k	number of cylinder
θ	dimensionless temperature	K	number of last cylinder in group
ρ	liquid density	χ	volume coefficient of filling
p	pressure	x, y, ξ	Cartesian coordinate systems
μ	dynamic viscosity of liquid	X_k, Y_k	coordinates of axes of cylinder
κ	thermal conductivity	$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\varphi},\ z^* = x - \mathrm{i}y = r\mathrm{e}^{-\mathrm{i}\varphi}$	polar coordinates
c	specific heat	$Z_k = X_k + \mathrm{i}Y_k = R_n\mathrm{e}^{\mathrm{i}\phi_k},\ Z_k^* = X_k - \mathrm{i}Y_k = R_n\mathrm{e}^{-\mathrm{i}\phi_k}$	polar coordinates of axis of cylinders
a	cylinder radius	F_{II}/S	permeability
b	pitch in cylinder array	α	coefficient of heat transfer
ε	ratio of cylinder radius to pitch in cylinder array	q	stream heat on surface
S	cross section falling on one cylinder	Nu	Nusselt number
S_0	dimensionless surfaces of basic central cell		

solutions: at first fields of the velocity and then fields of temperatures, at the flow round a regular arrangement of very thin cylinder, taking into consideration the heat of transfer. After calculating local functions of the velocity and the temperature we will consider global flow-thermal characteristics, which means the filter permeability and for the heat transfer between liquid and cylinders. The purpose of the paper is both to discuss the practised method and to introduce results describing processes of exchange for individual cases of nets: triangular, square, hexagonal and octagonal–square.

2. Problem statement

We consider the steady laminar flow of a viscous and a heat conducting fluid in the space along and between the cylinders with the velocity $\boldsymbol{w}$. The cylinders are identical and thin of the diameter $2a$ arranged regularly in distances b in such a way that points of breaking the perpendicular plane through them with their axes are arranged on vertexes of regular polygons (Fig. 1).

Along the cylinders the temperature of their surface changes linearly and there appears stationary heat transfer to the surrounding liquid. Arrangements of cylinders identically burdened dynamically and thermally are examined. We assume that in the liquid the density ρ, the specific heat capacity c, the coefficients: of the viscosity μ and the thermal conductivity κ, and the along components of gradients of the temperature T and pressures p are constants. In a cross section the temperature on the surface of cylinders is also fixed. The tenuousness of cylinders can be established by introducing the small parameter

$$\varepsilon = \frac{a}{b} \ll 1, \tag{1}$$

which we will carry out towards assessing, rejecting the small of higher order.

Let us introduce the Cartesian coordinate systems of x, y, ξ located symmetrically between cylinders with the axis

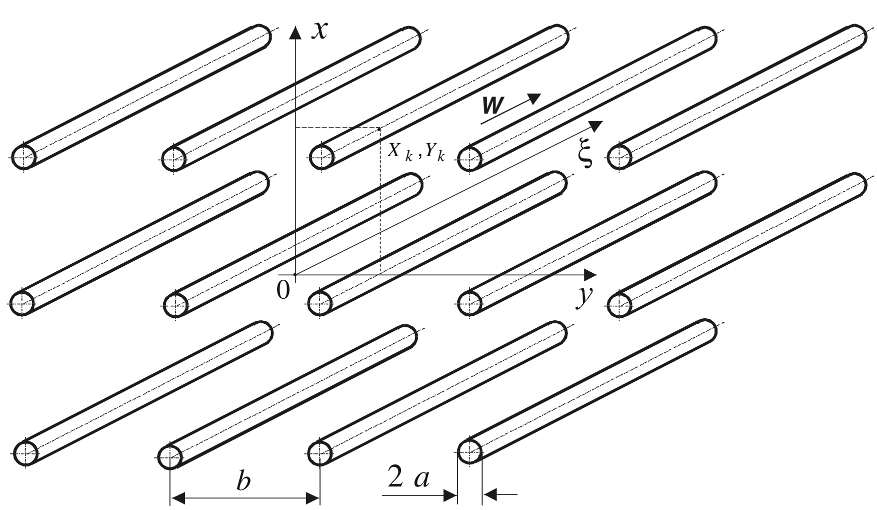


Fig. 1. Arrangement of cylinders in a space.

ξ parallel to the cylinder axes and we will mark as $X_k, Y_k \quad (k = 1, 2, \ldots)$ coordinates of the axis of cylinders in cross-section. We will still be treating coordinates x, y as dimensionless, for which the distance between cylinders b is the appropriate characteristic length scale. Apart from a two-dimensional Cartesian system of reference x, y at the same time we will have polar coordinates (Fig. 2) in harmony with the model in the complex numbers

$$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\varphi}, \quad z^* = x - \mathrm{i}y = r\mathrm{e}^{-\mathrm{i}\varphi}. \tag{2}$$

For the configurations of regularly arranged cylinders examined, parameters of the flow will be periodic functions x and y. At the selection of frames of reference of cylinders, we will be guided by criteria of the symmetry and selection of the element of a set of cells, the central cell, in which we will be determining the fields of the velocity and temperatures. The middle of the coordinate system x, y will be put in the centre of the central cell being the section with shapely polygon created from points of breaking the plain $\xi = 0$ with axes closest to the neighboring cylinders. If for the examined configuration of the arrangement of cylinders a few basic cells exist with different schedules of the velocity and temperatures of liquid, then we will alternatively establish centres of the right Cartesian agreements in centres of all of these cells.

For both the central cell and the coordinate system chosen, we will group cylinders according to their distance from the centre of symmetry, indicating the number of cylinder by k and through n the number of the next group of cylinders according to the assignment:

k	$1, 2, \ldots, K_0$	$k_1, k_1 + 1, \ldots K_1$	–	$k_n, k_n + 1, \ldots, K_n$
m_n	$m_0 = K_0$	$m_1 = K_1 - K_0$	–	$m_n = K_n - K_{n-1}$
n	0	1	–	n

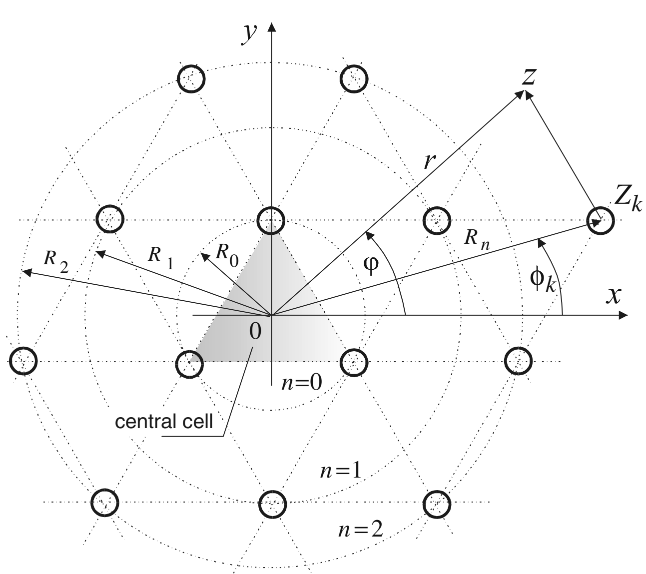


Fig. 2. Arrangement of cylinders in transverse plane.

where m_n is a number of cylinders contained in the n group and $k_n = K_{n-1} + 1$ is the first cylinder in the $n = 0$ group, in addition $k_0 = 1$. Through $n = N$ we will determine the amount of groups held for calculations. We will introduce two-dimensional coordinates of the axis of cylinders in the united record as follows (Fig. 2):

$$Z_k = X_k + \mathrm{i}Y_k = R_n\mathrm{e}^{\mathrm{i}\phi_k}. \tag{3}$$

Outlining the field of the velocity and temperatures in the central cell surrounded with cylinders belonging to the group $n = 0$ for the surface S_0 and the radius R_0 definite with the formula

$$S_0 = \frac{m_0}{4} ctg\frac{\pi}{m_0} - \left(\frac{m_0}{2} - 1\right)\pi\varepsilon^2, \quad R_0 = \left(2\sin\frac{\pi}{m_0}\right)^{-1} \tag{4}$$

will be a basic problem of this paper. Because of the periodic arrangement of cylinders, the distribution of parameters of the flow has also periodic character and therefore outlining these parameters in basic central cells will be sufficient. Outside dimensionless surfaces S_0 of basic central cells, we will still introduce the surface of a cross section S, falling on average to one cylinder. With its help, we can determine the volume coefficient of filling

$$\chi = \frac{\pi b^2}{S}\varepsilon^2. \tag{5}$$

Sizes $S, \chi/\varepsilon^2, S_0$ for essential configurations of cylinders are given in Table 1. Field of the velocity $\boldsymbol{w}$ and the temperature T will be introduced with the help of the dimensionless functions W and θ in the form:

$$T = \frac{\partial T}{\partial \xi}\left[\xi - \frac{b^4\rho c}{\mu\kappa}\frac{\partial p}{\partial \xi}\theta(x, y)\right] \quad \text{and}$$

$$\boldsymbol{w} = -\frac{b^2}{\mu}\frac{\partial p}{\partial \xi}W(x, y). \tag{6}$$

By substituting the above expressions into the basic equations of both the momentum and the energy, we will receive the system of two Poisson's equations in the same order:

$$\frac{\partial^2 W(x, y)}{\partial x^2} + \frac{\partial^2 W(x, y)}{\partial y^2} = -1, \tag{7}$$

$$\frac{\partial^2 \theta(x, y)}{\partial x^2} + \frac{\partial^2 \theta(x, y)}{\partial y^2} = W(x, y). \tag{8}$$

On the outlines of cylinders surrounding the area of the flow, the velocity and the temperature should fulfill the conditions

$$W_{|z_k|=\varepsilon} = 0 \quad \text{and} \tag{9}$$

$$\theta_{|z_k|=\varepsilon} = 0, \quad z_k = z - Z_k, \quad k = 1, 2, \ldots \tag{10}$$

In this way the problem posed was led, in turn, to at first determining the field of the velocity W from the Eq. (7) with the boundary conditions (9) and then fields of temperatures θ from the Eq. (8) with the boundary conditions (10) given. As can be seen, for solving the second problem the need to determine the field of the velocity exists. One

Table 1
Geometrical parameters and integrals by Eq. (11) for different nets

	n	m_n	K_n	R_n	$tg\phi_{k_n}$	Variant	A_n	$\sum_{k=k_0}^{K_n} \int\int_{S_0} W_k \mathrm{d}S$	$\int\int_{S_0} W_n \mathrm{d}S$	$\sum_{n=0_0}^{N_n} \int\int_{S_0} W_n \mathrm{d}S$
Square array	0	4	4	$\frac{\sqrt{2}}{2}$	1	*a*	−0.6931	−2.9443	−3.6374	−3.6374
	1	8	12	$\frac{\sqrt{10}}{2}$	$\frac{1}{3}$	*b*	−7.3777	7.3334	−0.0443	−3.6817
	2	4	16	$\frac{3\sqrt{2}}{2}$	1	*a*	−5.9915	6.0164	0.0248	−3.6569
	3	8	24	$\sqrt{\frac{13}{2}}$	$\frac{1}{5}$	*b*	−14.9811	14.9738	−0.0073	−3.6642
Triangular array	0	3	3	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	*a*	0	−1.5409	−1.5409	−1.5409
	1	3	6	$\frac{2\sqrt{5}}{3}$	0	*a*	−1.0986	0.3843	−0.0914	−1.6323
	2	6	12	$\sqrt{\frac{7}{3}}$	$\frac{\sqrt{3}}{5}$	*b*	−4.9698	2.1963	0.0493	−1.5830
	3	6	18	$\sqrt{\frac{13}{3}}$	$\frac{\sqrt{3}}{7}$	*b*	−8.7980	3.8096	−0.0001	−1.5831
Hexagonal array	0	6	6	1	$\frac{\sqrt{3}}{3}$	*a*	−3.5835	−0.2338	−9.5440	−9.5440
	1	6	12	2	$\frac{\sqrt{3}}{3}$	*a*	−8.2963	21.6102	0.0818	−9.4622
	2	12	24	$\sqrt{7}$	$\frac{\sqrt{3}}{9}$	*b*	−23.3509	60.6675	0.0001	−9.4621
Octagon–square array	0	4	4	$\frac{\sqrt{2}}{2}$	1	*a*	−0.6931	−2.9443	−3.6374	−3.6374
	1	4	8	$1+\frac{\sqrt{2}}{2}$	1	*a*	−4.2186	4.2784	0.0598	−3.5776
	2	8	16	$\frac{\sqrt{14+5\sqrt{2}}}{2}$	$\frac{1+2\sqrt{2}}{7}$	*b*	−14.7498	14.7601	0.0103	−3.5673
	5	4	20	$\frac{3\sqrt{2}}{2}+1$	1	*a*	−9.1007	9.1060	0.0053	−3.5620

should pay attention to the fact that Eqs. (7) and (8) are linear and the boundary conditions (9) and (10) are homogeneous. For solving the problem posed, it is possible to apply rules of the superposition.

2.1. Method of determining the velocity field

The field sought of the longitudinal velocity is defined with the help of dimensionless functions $W(x,y)$ to accomplish putting Poisson on the same level (7) and the boundary conditions (9) of zeroing oneself on the surface of the velocity of fixed cylinders. We will be seeking an approximate solution of the sum, which will fulfill the boundary conditions (9) only for small values ε on the limited number K_N of cylinders, in the form satisfying the Eq. (7)

$$W(x,y;\varepsilon,N) = \hat{W}(x,y) + \frac{1}{4\pi}\frac{S}{b^2}\left[A(\varepsilon) + \sum_{n=0}^{N} W_n(x,y)\right], \tag{11}$$

where

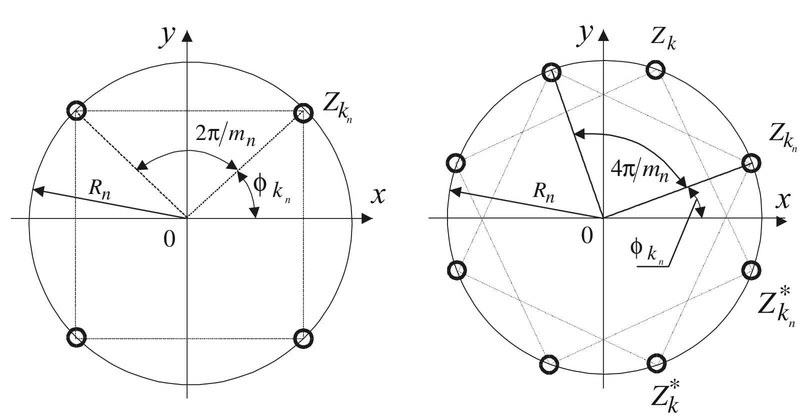
$$\hat{W}(x,y) = \frac{R_0^2 - (x^2+y^2)}{4} = \frac{R_0^2 - zz^*}{4} = \frac{R_0^2 - r^2}{4} \tag{12}$$

is the special solution of Poisson's equation (5),

$$W_n(x,y) = A_n + B_n \sum_{k=k_n}^{K_n} \ln(z - Z_k)(z - Z_k)^* \tag{13}$$

is the solution of the Laplace equation relating to $m_n = K_n - k_n + 1$ straight poles put in axes of cylinders on a circle of the radius R_n, and $A(\varepsilon)$, A_n, B_n are constants meanwhile not determined.

Using symmetrical arrangement of points Z_k, we can describe the post W_n in a more comfortable form. These points are arranged in examined regular nets either like in the variant (a) on vortexes of one regular polygon or like in the variant (b) on tops of two identical regular polygons of the symmetrical location relative to the axis of coordinates (Fig. 3). For the variant (a)



variant(a) variant (b)

Fig. 3. Two variants arrangements of cylinders in cells.

$$Z_k = Z_{kn} \cdot e^{i\frac{2\pi}{m_n}(k-k_n)}, \quad k_n \leqslant k \leqslant K_n, \quad n = 0, 1, \dots$$

we can convert sums appearing in (13) according to formula resulting from algebra

$$\begin{aligned}\sum_{k=k_n}^{K_n} \ln(z - Z_k)(z - Z_k)^* &= \ln \prod_{k_n}^{K_n} (z - Z_k)(z - Z_k)^* = \ln(z^{m_n} - Z_{k_n}^{m_n})(z^{m_n} - Z_{k_n}^{m_n})^* \\ &= \ln R_n^{2m_n}\left[1 - 2\left(\frac{r}{R_n}\right)^{m_n} \cos m_n(\varphi - \phi_{k_n}) + \left(\frac{r}{R_n}\right)^{2m_n}\right].\end{aligned} \tag{14}$$

For the variant (b)

$$Z_k = \begin{cases} Z_{k_n} e^{i\frac{4\pi}{m_n}(k-k_n)}, & k_n \leqslant k \leqslant k_n + \frac{m_n}{2} - 1 \\ Z_{k_n} e^{i\frac{4\pi}{m_n}(K_n-k)}, & k_n + \frac{m_n}{2} \leqslant k \leqslant K_n \end{cases}$$

the half from the total of points in the given group is a mirror reflection with respect to an axis of symmetry x of remained points. So we can transform series (13) as follows

$$\begin{aligned}\sum_{k_n}^{K_n} \ln(z - Z_k)(z - Z_k)^* &= \sum_{k_n}^{k_n + \frac{m_n}{2} - 1} \ln(z - Z_k)(z - Z_k)^* + \sum_{k_n}^{K_n} \ln(z - Z_k)(z - Z_k^*)^* \\ &= \ln R_n^{2m_n}\left[1 - 2\left(\frac{r}{R_n}\right)^{\frac{m_n}{2}} \cos \frac{m_n}{2}(\varphi - \phi_{k_n}) - \left(\frac{r}{R_n}\right)^{m_n}\right] \\ &\quad \times \left[1 - 2\left(\frac{r}{R_n}\right)^{\frac{m_n}{2}} \cos \frac{m_n}{2}(\varphi + \phi_{k_n}) + \left(\frac{r}{R_n}\right)^{m_n}\right].\end{aligned} \tag{15}$$

Sum $W(x, y; \varepsilon, N)$ (11) satisfies the Navier–Stokes equation closely, which is reduced, in the examined case, to the Poisson Eq. (7), and we will try to select constants $A(\varepsilon), A_n, B_n$ appearing in it, in order the boundary conditions (9) be fulfilled possibly best on account of the small parameter ε and on possibly the largest number of cylinders surrounding the central cell. Accurate fulfilling, by the established figure of the solution (11) of condition (9) on all cylinders does not even seem possible or intentional, and therefore, for the finished number of cylinders K_n, we will be selecting constants $A(\varepsilon), A_n, B_n$ in such a way so that the mistake of the gotten approximation inside the central cell does not grow with ε faster than linearly.

Exploiting the condition of the balance of forces on every cylinder

$$\mu \int_0^{2\pi} \frac{\partial \boldsymbol{w}}{\partial r_k} r_k \mathrm{d}\varphi = -S \frac{\partial p}{\partial \xi}, \quad r_k = |z - Z_k| = \varepsilon$$

with the accuracy to terms of the zero order for the sake on ε, we receive

$$B_n = 1, \quad n = 0, 1, 2, \dots \tag{16}$$

It is possible now to introduce condition $W = 0$ on any k cylinder $z - Z_k = \varepsilon e^{i\psi}$ in the form

$$\begin{aligned}0 = \frac{R_0^2}{4} - \frac{Z_k Z_k^* + \varepsilon(Z_k e^{-i\psi} + Z_k^* e^{i\psi}) + \varepsilon^2}{4} \\ + \frac{S}{4\pi b^2}\Bigg\{A(\varepsilon) + \sum_{n=0}^{N} A_n + \sum_{l=1}^{K_n} \ln[(Z_k - Z_l)(Z_k - Z_l)^* \\ + \varepsilon((Z_k - Z_l)e^{-i\psi} + (Z_k - Z_l)^* e^{i\psi}) + \varepsilon^2]\Bigg\}\end{aligned} \tag{17}$$

Let us notice that one of the expressions for the sign $l = k$ for $\varepsilon \to 0$ is becoming odd and in order to remove the singularity, we will introduce

$$A(\varepsilon) = -\ln \varepsilon^2. \tag{18}$$

The above relation permits to fulfill the condition (17) on every cylinder with the accuracy to terms of the lowest order, thinned out from the consideration on ε. However accepting

$$\begin{aligned}A_0 &= -\sum_{l=2}^{K_0} \ln(Z_1 - Z_l)(Z_1 - Z_l)^* = -2\ln(m_0 R_0^{m_0-1}) \quad n = 0 \\ A_n &= -\sum_{l=k_n}^{K_n} \ln(Z_1 - Z_l)(Z_1 - Z_l)^* = -2m_n \ln R_n - \\ &\quad - \begin{cases} \ln\left[1 - 2\left(\frac{R_0}{R_n}\right)^{m_n} \cos m_n(\phi_1 - \phi_{k_n}) + \left(\frac{R_0}{R_n}\right)^{2m_n}\right] \; (a) \\ \ln\left[1 - 2\left(\frac{R_0}{R_n}\right)^{\frac{m_n}{2}} \cos \frac{m_n}{2}(\phi_1 - \phi_{k_n}) + \left(\frac{R_0}{R_n}\right)^{m_n}\right] \\ \left[1 - 2\left(\frac{R_0}{R_n}\right)^{\frac{m_n}{2}} \cos \frac{m_n}{2}(\phi_1 + \phi_{k_n}) + \left(\frac{R_0}{R_n}\right)^{m_n}\right] \; (b) \end{cases} \quad n = 1, 2, \dots\end{aligned} \tag{19}$$

we will fulfill conditions (17) on cylinders of the central cell $n = 0$ with the accuracy to two terms of the lowest order for the sake of ε. Using the expressions (14), (15), (17)–(19) we will get W_n(13) in the complex notation to the form

$$W_n = \left\{\begin{array}{ll} \ln\frac{R_0^2}{m_0^2} + \ln\left[1 - \left(\frac{z}{Z_1}\right)^{m_0}\right]\left[1 - \left(\frac{z^*}{Z_1^*}\right)^{m_0}\right] & n = 0 \\ \ln\dfrac{\left[1 - \left(\frac{z}{Z_{k_n}}\right)^{m_n}\right]\left[1 - \left(\frac{z^*}{Z_{k_n}^*}\right)^{m_n}\right]}{\left[1 - \left(\frac{Z_1}{Z_{k_n}}\right)^{m_n}\right]\left[1 - \left(\frac{Z_1^*}{Z_{k_n}^*}\right)^{m_n}\right]} \; (a) & \\ \ln\dfrac{\left[1 - \left(\frac{z}{Z_{k_n}}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{z^*}{Z_{k_n}^*}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{z}{Z_{k_n}^*}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{z^*}{Z_{k_n}}\right)^{\frac{m_n}{2}}\right]}{\left[1 - \left(\frac{Z_1}{Z_{k_n}}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{Z_1^*}{Z_{k_n}^*}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{Z_1}{Z_{k_n}^*}\right)^{\frac{m_n}{2}}\right]\left[1 - \left(\frac{Z_1^*}{Z_{k_n}}\right)^{\frac{m_n}{2}}\right]} \; (b) & \end{array}\right\} \quad n = 1, 2, \dots \tag{20}$$

Using (12)–(14), (16)–(19), we receive the approximate solution (11) in the polar coordinates r, φ in the form

$$W(r,\varphi;\varepsilon,N) = \frac{R_0^2 - r^2}{4} + \frac{S}{4\pi b^2}\left[-\ln\varepsilon^2 + \sum_{n=0}^{N} W_n(r,\varphi)\right], \tag{21}$$

where

$$W_n = \left\{\begin{array}{ll} \ln\frac{R_0^2}{m_0^2} + \ln\left[1 - 2\left(\frac{r}{R_0}\right)^{m_0}\cos m_0(\varphi-\phi_1) + \left(\frac{r}{R_0}\right)^{2m_0}\right] & n=0 \\ \ln\dfrac{\left[1-2\left(\frac{r}{R_n}\right)^{m_n}\cos m_n\left(\varphi-\phi_{k_n}\right)+\left(\frac{r}{R_0}\right)^{2m_n}\right]}{\left[1-2\left(\frac{R_0}{R_n}\right)^{m_n}\cos m_n\left(\phi_1-\phi_{k_n}\right)+\left(\frac{R_0}{R_n}\right)^{2m_n}\right]} \quad (a) & \\ \ln\dfrac{\left[1-2\left(\frac{r}{R_n}\right)^{\frac{m_n}{2}}\cos\frac{m_n}{2}\left(\varphi-\phi_{k_n}\right)+\left(\frac{r}{R_n}\right)^{m_n}\right]\left[1-2\left(\frac{r}{R_n}\right)^{\frac{m_n}{2}}\cos\frac{m_n}{2}\left(\varphi+\phi_{k_n}\right)+\left(\frac{r}{R_n}\right)^{m_n}\right]}{\left[1-2\left(\frac{R_0}{R_n}\right)^{\frac{m_n}{2}}\cos\frac{m_n}{2}\left(\phi_1-\phi_{k_n}\right)+\left(\frac{R_0}{R_n}\right)^{m_n}\right]\left[1-2\left(\frac{R_0}{R_n}\right)^{\frac{m_n}{2}}\cos\frac{m_n}{2}\left(\phi_1+\phi_{k_n}\right)+\left(\frac{R_0}{R_n}\right)^{m_n}\right]} \quad (b) & \end{array}\right\} n = 1,2,\ldots \tag{22}$$

This solution fulfills the Poisson equation closely (7), but the boundary conditions (9) are only approximately fulfilled on cylinders of the central cell with the misconception being able to grow linearly with ε.

Formulas (21) and (22) present results received in the more comfortable notation than depicted in [10]. Thus obtained distributions of the velocity for a few ε and N are shown for the square net [10], and for the triangular and hexagonal nets in [11].

Knowing the distribution of the velocity in central cells we can determine the average velocity of the flow in each of these cells

$$\bar{\boldsymbol{w}} = -\frac{b^2}{\mu}\frac{\partial p}{\partial \xi}\bar{W}, \quad \bar{W} = \frac{1}{S_0}\int\int_{S_0} W(r,\varphi) r\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{23}$$

Calculations of the integral $\int\int_{S_0} W(r,\varphi) r\,\mathrm{d}r\,\mathrm{d}\varphi$ implicating the longitudinal permeability for different nets are presented in Tables 1 and 2 in Fig. 4.

2.2. Method of determining the temperature field

The sought field of the temperature is determined with the help of the dimensionless function $\theta(x,y)$ satisfying the Poisson Eq. (8) and the boundary conditions (19) of zeroing the temperature on the surface of cylinders. Changeable right side of the Eq. (8) $W(x,y;\varepsilon,N)$ has been defined by formula (21) in the previous part of the paper.

We will be seeking the approximate solution of this task in the sum

$$\theta(x,y;\varepsilon,N) = \hat{\theta}(x,y) + \frac{S}{4\pi b^2}\left[C(\varepsilon) + \sum_{n=0}^{N}(\hat{\theta}_n + \theta_n)\right], \tag{24}$$

where

$$\hat{\theta}(x,y) = \frac{R_0^2 - r^2}{64}(r^2 - 3R_0^2) + \frac{S}{4\pi b^2}A(\varepsilon)\frac{r^2 - R_0^2}{4} \tag{25}$$

and

$$\hat{\theta}_n(x,y) = A_n\frac{r^2 - R_0^2}{4} + \sum_{k=k_n}^{K_n}\frac{(z - Z_k)(z - Z_k)^*}{4} \times [\ln(z - Z_k)(z - Z_k)^* - 2] + C_n \tag{26}$$

they are special solutions of the Poisson equations:

$$\frac{\partial^2\hat{\theta}}{\partial x^2} + \frac{\partial^2\hat{\theta}}{\partial y^2} = \hat{W}(x,y) + \frac{S}{4\pi b^2}A(\varepsilon), \tag{27}$$

and

$$\frac{\partial^2\hat{\theta}_n}{\partial x^2} + \frac{\partial^2\hat{\theta}_n}{\partial y^2} = \hat{W}(x,y). \tag{28}$$

The expression

$$\theta_n = D_n + E_n\sum_{k=k_n}^{K_n}\ln(z - Z_k)(z - Z_k)^*, \tag{29}$$

Table 2
Longitudinal permeability for different nets

Kind of net	$\frac{S}{b^2}$	S_0	$\frac{\chi}{\varepsilon^2}$	$\int\int_{S_0}\hat{W}\mathrm{d}S$	$\frac{F_{II}}{S}$
Square array	1	1	π	$\frac{1}{12} = 0.0833$	$\frac{1}{4\pi}\left[\ln\frac{1}{\chi} - 1.4723\right]$
Triangular array	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{2\pi}{\sqrt{3}}$	$\frac{\sqrt{3}}{64} = 0.0271$	$\frac{1}{4\pi}\left[\ln\frac{1}{\chi} - 1.4718\right]$
Hexagonal array	$\frac{3\sqrt{3}}{4}$	$\frac{3\sqrt{3}}{2}$	$\frac{4\pi}{3\sqrt{3}}$	$\frac{7\sqrt{3}}{32} = 0.3789$	$\frac{1}{4\pi}\left[\ln\frac{1}{\chi} - 1.3481\right]$
Octagon–square array	$\frac{3+2\sqrt{2}}{4}$	$2(1+\sqrt{2})$	$\frac{4\pi}{3+2\sqrt{2}}$	$\frac{10\sqrt{2}+13}{24} = 1.1309$	$\frac{1}{4\pi}\left[\ln\frac{1}{\chi} - 0.9713\right]$
Permeability average	$\frac{F_{II}}{S} = \frac{1}{4\pi}\left[\ln\frac{1}{\chi} - 1.1596\right]$				

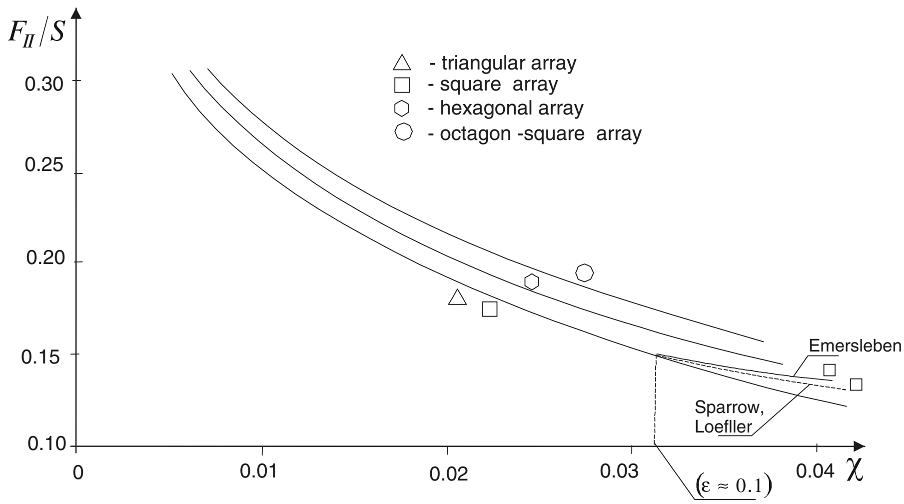


Fig. 4. Dependence of longitudinal permeability F_{II}/S on filling χ.

is the solution of the Laplace equation of cylinders on m_n straight poles placed in the axe cylinders belonging to the group n. The constants A_n are determined by Eq. (19). Constants $C(\varepsilon), C_n, D_n$ and E_n must be so well-matched that the mistake of the gotten approximation is possibly smallest.

The solutions (24) satisfy closely the Poisson Eq. (8), but cannot exactly fulfill the boundary conditions (10). We will try to establish unknown constants $C(\varepsilon), C_n, D_n$ and E_n so that the balance of the energy and the boundary conditions on the surface of cylinders of the central cell were fulfilled approximately (10).

Using the condition of the conservation of energy on every cylinder for the element of the net

$$-\kappa \int_0^{2\pi} \frac{\partial T}{\partial r_k} r_k \mathrm{d}\varphi = \rho c \frac{\partial T}{\partial \xi} \bar{\mathbf{w}} S \tag{30}$$

with the accuracy to terms of the zero order for the sake of ε, we will receive

$$E_n = \bar{W}, \quad n = 0, 1, 2, \ldots. \tag{31}$$

Condition $\theta = 0$ on the cylinder of the central cell $|z - Z_k| = \varepsilon$ (for the sake of the law of the symmetry the condition is satisfied automatically on remained cylinders of the central cell) can be presented introducing the constants, in the form:

$$C(\varepsilon) = -\bar{W} \ln \varepsilon^2, \tag{32}$$

$$C_0 = -\sum_{k=1}^{m_0} \frac{(Z_1 - Z_k)(Z_1 - Z_k)^*}{4} \ln(Z_1 - Z_k)(Z_1 - Z_k)^* + m_0 R_0^2, \qquad n = 0$$

$$C_n = -\sum_{k=k_n}^{K_{n0}} \frac{(Z_1 - Z_k)(Z_1 - Z_k)^*}{4} \ln(Z_1 - Z_k)(Z_1 - Z_k)^* + \frac{m_n}{2}(R_0^2 + R_n^2), \qquad n = 1, 2, \ldots \tag{33}$$

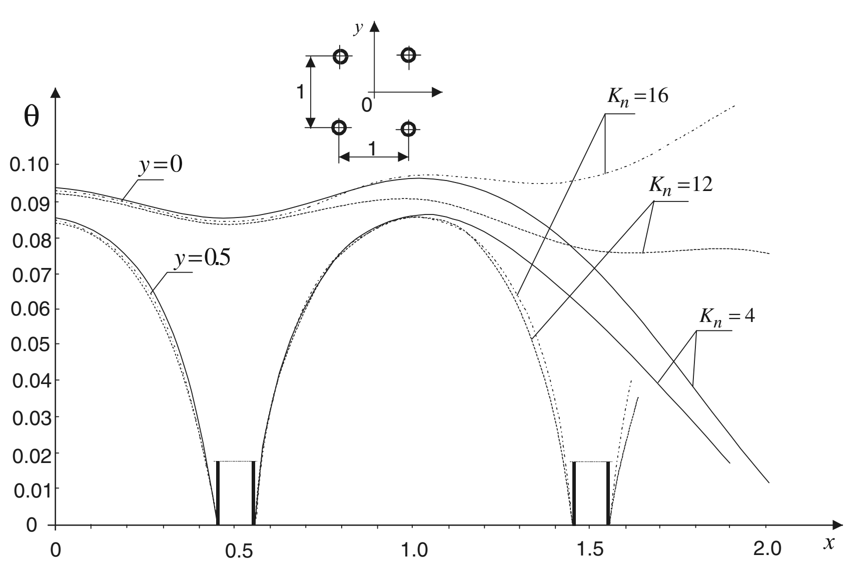


Fig. 5. Distribution of temperature for square array ($\varepsilon = 0.05$).

and

$$D_0 = -\bar{W}\sum_{k=1}^{m_0} \ln(Z_1 - Z_k)(Z_1 - Z_k)^*, \quad n = 0$$
$$D_n = -\bar{W}\sum_{k=k_n}^{K_{n0}} \ln(Z_1 - Z_k)(Z_1 - Z_k)^*, \quad n = 1,2,\ldots \tag{34}$$

We fulfill the mentioned above condition for ($\theta = 0$) with the accuracy to terms of the order $\varepsilon \ln \varepsilon$.

Using the expressions (20), (25), (26), (29), (32), (33), (34), we receive the approximate solution (24) in polar coordinates r, φ in the form

$$\theta(r,\varphi;\varepsilon,N) = \frac{R_0^2 - r^2}{64}(r^2 - 3R_0^2) + \left[\left(\frac{R_0^2 - r^2}{4} - \bar{W}\right)\ln\varepsilon^2 + \sum_{n=0}^{N}(\hat{\theta}_n + \theta_n)\right], \tag{35}$$

where

$$\begin{aligned}\hat{\theta}_0 = {} & A_0\frac{r^2 - R_0^2}{4} + R_0^2\sum_{k=1}^{m_0}\left\{\left[1 - 2\frac{r}{R_0}\cos(\varphi - \phi_k) + \frac{r^2}{R_0^2}\right]\right. \\ & \times \ln R_0^2\left[1 - 2\frac{r}{R_0}\cos(\varphi - \phi_k) + \frac{r^2}{R_0^2}\right] \\ & - [2 - 2\cos(\phi_1 - \phi_k)]\ln R_0^2[2 - 2\cos(\phi_1 - \phi_k)] \\ & - \frac{m_0}{2}(r^2 - R_0^2), \quad n = 0, \\ \hat{\theta}_n = {} & A_n\frac{r^2 - R_0^2}{4} + R_n^2\sum_{k=k_n}^{K_n}\left\{\left[1 - 2\frac{r}{R_n}\cos(\varphi - \phi_k) + \frac{r^2}{R_n^2}\right]\right. \\ & \times \ln R_n^2\left[1 - 2\frac{r}{R_n}\cos(\varphi - \phi_k) + \frac{r^2}{R_n^2}\right] \\ & - \left[2 - 2\frac{R_0}{R_n}\cos(\phi_1 - \phi_k) + \frac{R_0^2}{R_n^2}\right] \\ & \times \ln R_n^2\left[2 - 2\frac{R_0}{R_n}\cos(\phi_1 - \phi_k) + \frac{R_0^2}{R_n^2}\right] \\ & - \frac{m_n}{2}(r^2 - R_0^2), \quad n = 1,2,\ldots\end{aligned} \tag{36}$$

and

$$\theta_n = \bar{W}W_n \quad n = 0,1,2,\ldots. \tag{37}$$

Example of the distribution of the temperature (35) for the square net is shown in Fig. 5. As can be seen, like at the distribution of the velocity [10,11], for not very thick cylinders, the received result seems satisfactory even when cylinders taken into consideration surrounding the central cell are not many.

Knowing $\theta(x,y)$, we can determine the average temperature

$$\bar{T} = \frac{\partial T}{\partial\xi}\left[\xi - \frac{b^4\rho c}{\mu\kappa}\frac{\partial p}{\partial\xi}\bar{\theta}\right], \quad \bar{\theta} = \frac{\int\int_{S_0}\theta W r\mathrm{d}r\mathrm{d}\varphi}{\int\int_{S_0} W r\mathrm{d}r\mathrm{d}\varphi}, \tag{38}$$

and the average stream heat on the surface of the cylinder

$$\bar{q} = -\kappa\frac{\partial T}{\partial r_k}\bigg|_{r_k=\varepsilon} = \frac{b^4\rho c}{2\pi a\mu}\frac{\partial T}{\partial\xi}\frac{\partial p}{\partial\xi}\int_0^{2\pi}\frac{\partial\theta}{\partial r_k}r_k|_{r_k=\varepsilon}\mathrm{d}\varphi. \tag{39}$$

Introducing the coefficient of heat transferα between cylinders and liquid in harmony with the model

$$\bar{q} = \alpha(T_{r=a} - \bar{T}),$$

and Nusselt number $Nu = \alpha\sqrt{S}/\kappa$ after taking into consideration (23), (38), (39) and for using the condition of the balance sheet of the energy for the element of the net (30), we receive

$$Nu = \frac{1}{2\sqrt{\pi\chi}}\frac{\bar{W}}{\bar{\theta}}\frac{S}{b^2} \tag{40}$$

from here, substituting the approximation values $\bar{W}$ and $\bar{\theta}$, and rejecting the little magnitudes of the higher order, we receive the expression

$$Nu = \frac{2\sqrt{\frac{\pi}{\chi}}}{\ln\frac{\pi}{\chi} - \ln\frac{S}{b^2}} + \cdots \tag{41}$$

3. Conclusions

The main concern in this paper is determination of the along flow of the parallel cylinders regularly arranged and heat transfer between the cylinders and the fluid flow. Because hydrodynamic conditions have the deciding influence on the course of the thermal phenomenon occurrences here, therefore for examining the heat transfer between cylinders and surrounding liquid, there is a need first to determine the flow of this liquid. That results from the fact that in liquid both the field of the velocity and field of temperatures should satisfy the same Poisson equation but with different right sides, where for the temperature on the right one there is a function describing the distribution of the velocity, which previously was determined.

To receive the approximate solution of the posed problem a method of limiting itself for using the finished number of sufficiently thin cylinders arranged symmetrically around the central cell inside which the distribution of parameters of the flow being appointed was applied. It is possible here to reduce the three-dimensional problem leading to the two-dimensional task, for which, for carrying calculations, it was comfortable to use extended complex plane. The distribution of the velocity and temperatures satisfy the equations of flow and the heat transfer strictly, however on surfaces of cylinders the boundary conditions are satisfied approximately, with the dependent misconception in the not very essential way both from the number of cylinders taken into consideration as well as, above all, from the ratio of their thickness to the distance between them. For sufficiently thin cylinders the mistakes are not considerable.

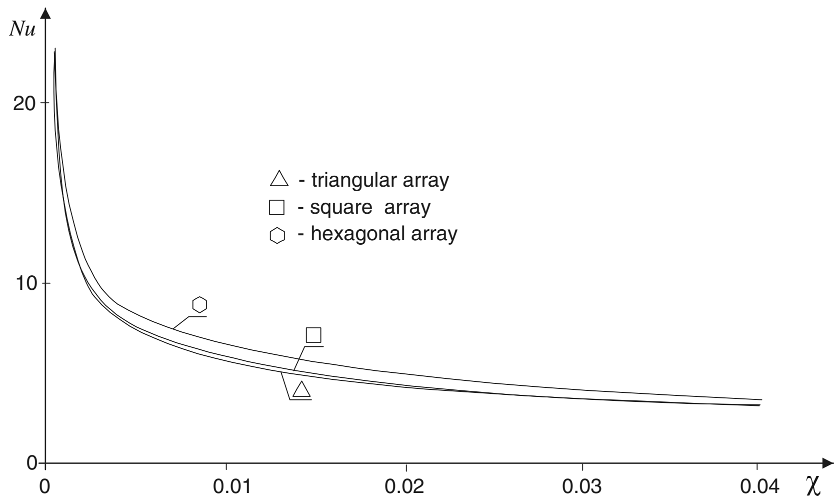


Fig. 6. Dependent Nusselt number Nu on filling χ.

Apart from the analysis of the flow and the heat transfer on the local scale, for the local distribution of the velocity and the temperature, characterizations on the global scale for the hydrodynamic permeability and for the global heat transfer between cylinders and surrounding liquid deserve attention.

Comparing research findings of the longitudinal permeability, for arrangements of cylindrical rods we could become convinced that it depends very poorly on the geometrical configuration of arranging cylinders. If we introduce, in the Darcy law, the dimensionless coefficient of the longitudinal permeability F_{II}/S for the average surface S for one cylinder, we find that this coefficient strongly depends on the filling χ, however, for different distributions of cylinders, its different values do not differ between themselves in the essential way (see Fig. 4 and Table 2). The question arises whether the global heat transfer does not have similar properties.

The global of the heat transfer is determined by the Nusselt number, for which the surface S is the square of the characteristic number. The results received are presented in the graphs form in Fig. 6. And it occurs that for the system of cylinders, the Nusselt number Nu describing the heat transfer depends strongly only on the coefficient of the filling χ and poorly the longitudinal permeability depending on the configuration of arranging cylinders. Graphs shown in Figs. 4 and 6 for a few regular configurations of cylinders suggest conjecture that the coefficient filling χ is the most essential parameter determining the global flow-thermal characterizations for the flow of liquid along the arrangement of parallel cylinders but the way of arranging these cylinders plays the secondary role. Since of these characteristics for different systems is very difficult to create, therefore the use of approximation results received for a simple configuration may possibly be useful. It seems that for a few special regular systems the results received could be used also for other evenly satisfying systems though not necessarily for regularly arranged cylinders.

Acknowledgement

Let this work be an expression of my gratitude and respect for Professor Andrzej Szaniawski, for his participation in my scientific development.

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